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Steady flow of a micropolar fluid due to a rotating disc

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SUMMARY

Three-dimensional, axially-symmetric, steady flow of a micropolar fluid, due to a rotating disc, is considered. The resulting equations of motion are solved numerically, for four different combinations of the six parameters involved, using the Gauss-Seidel iterative procedure and Simpson's rule. Results are presented both in tabular and graphical form.

1. Introduction

Eringen [4] has presented the theory of micropolar fluids. The motion of a micropolar fluid can be described by a velocity vector $\boldsymbol{v}$ together with a local spin vector $\boldsymbol{\nu}$. The stress- and couple-stress tensors $\mathbf{t}$ and $\mathbf{m}$ are linear combinations of $\boldsymbol{\nu}$ and spatial derivatives of $\boldsymbol{v}$ and $\boldsymbol{\nu}$ involving six viscosity coefficients λ, μ, κ, α, β, γ as well as the density ρ and the micro-inertia coefficient j [4]. The micropolar fluid model has been used to describe the flow of liquid crystals [8] and blood flow [1, 2]. Guram and Smith [6] suggested a method of estimating the material constants, when the coupling constant is small.

Von Kármán [7] obtained an approximate solution to the problem of the rotating disc for Newtonian fluids, using momentum-integral methods. Cochran [3] found an accurate numerical solution to the same problem. He used expansions in powers of the dimensionless independent variable for the flow near the disc, and an expansion in powers of the exponential function for large values of the independent variable which were then joined together for intermediate values.

In this paper, we consider the steady flow of a micropolar fluid, due to a rotating disc. The basic equations of motion are given in Section 2. In Section 3, we reduce the equations of motion to seven ordinary differential equations in dimensionless form, involving six parameters, using dimensional analysis. The finite-difference equations are presented in Section 4. In Section 5, the numerical procedure is described. Results are given in tabular and graphical form and discussed in Section 6.

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2. Equations of motion

The equations of motion of a micropolar fluid, with isotropic structure, as given in [4], are

$$\dot{\rho} + \rho(\nabla \cdot v) = 0,$$
$$(\lambda + 2\mu + \kappa)\nabla(\nabla \cdot v) - (\mu + \kappa)\nabla \times \nabla \times v + \kappa \nabla \times \nu - \nabla p + \rho f = \rho \dot{v}, \tag{2.1}$$
$$(\alpha + \beta + \gamma)\nabla(\nabla \cdot \nu) - \gamma(\nabla \times \nabla \times \nu) + \kappa \nabla \times v - 2\kappa\nu + \rho \ell = \rho j \dot{\nu}$$

being, respectively, the conservation of mass, and balance of linear and angular momenta. v denotes velocity, ν the micro-rotation, *or* spin, p the thermodynamic pressure, f and ℓ the body-force and -couple per unit mass, ρ the density and j the micro-inertia; λ, μ, κ, α, β, and γ are material constants (viscosity coefficients) and the dot signifies the material differentiation with respect to time. Thermal effects have been neglected.

The constitutive equations giving $t_{k\ell}$ and $m_{k\ell}$, the stress and couple-stress tensors, in curvilinear co-ordinates, are

$$t_{k\ell} = (-p + \lambda v_{r;r}) g_{k\ell} + \mu(v_{k;\ell} + v_{\ell;k}) + \kappa(v_{\ell;k} - \epsilon_{k\ell r}\nu_r), \tag{2.2}$$
$$m_{k\ell} = \alpha\, \nu_{r;r}\, g_{k\ell} + \beta \nu_{k;\ell} + \gamma \nu_{\ell;k}$$

where $g_{k\ell}$ and $\epsilon_{k\ell r}$ are the metric tensor and the covariant ϵ-symbol, respectively; the semi-colon denotes covariant partial differentiation with respect to a space co-ordinate and the summation convention has been used.

The material constants must satisfy the following inequalites, derived from the Clausius – Duhem inequality:

$$3\lambda + 2\mu + \kappa \geq 0, \quad 2\mu + \kappa \geq 0, \quad \kappa \geq 0,$$
$$3\alpha + \beta + \gamma \geq 0, \quad \gamma \geq |\beta|. \tag{2.3}$$

3. Flow due to a rotating disc

We consider an infinite plane disc, rotating with angular velocity Ω in an otherwise unbounded micropolar fluid, at rest apart from the motion induced by the disc. We consider the motion of the micropolar fluid on the side of the plane for which z is positive. We take cylindrical polar co-ordinates r, θ, z with velocity components u, v, w and micro-rotation components ν_1, ν_2, ν_3, where $r = 0$ is the axis of rotation of the plane of the disc, $z = 0$ (see Figure 1).

The material constants of the micropolar fluid are assumed to be independent of position and we neglect body forces and body couples. We assume the flow is steady, incompressible, and axially-symmetric. We look for a solution which is independent of θ and t, and so we set

$$v = [u(r, z), v(r, z), w(r, z)], \nu = [\nu_1(r, z), \nu_2(r, z), \nu_3(r, z)]. \tag{3.1}$$

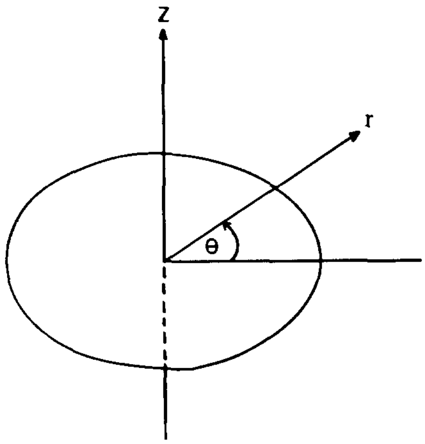

Figure 1. Co-ordinate system for rotating-disc flow

Substituting (3.1) in (2.1) we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0, \tag{3.2}$$

$$(\mu+\kappa)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right)-\kappa\frac{\partial \nu_2}{\partial z}-\frac{\partial p}{\partial r}=\rho\left(\frac{w\partial u}{\partial z}-\frac{v^2}{r}+u\frac{\partial u}{\partial r}\right), \tag{3.3}$$

$$(\mu+\kappa)\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right)+\kappa\left(\frac{\partial \nu_1}{\partial z}-\frac{\partial \nu_3}{\partial r}\right)=\rho\left(w\frac{\partial v}{\partial z}+\frac{uv}{r}+u\frac{\partial v}{\partial r}\right), \tag{3.4}$$

$$(\mu+\kappa)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)+\frac{\kappa}{r}\frac{\partial}{\partial r}(r\nu_2)-\frac{\partial p}{\partial z}=\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right), \tag{3.5}$$

$$(\alpha+\beta+\gamma)\frac{\partial}{\partial r}\left(\frac{\partial \nu_1}{\partial r}+\frac{\nu_1}{r}+\frac{\partial \nu_3}{\partial z}\right)-\gamma\frac{\partial}{\partial z}\left(\frac{\partial \nu_3}{\partial r}-\frac{\partial \nu_1}{\partial z}\right)-\kappa\frac{\partial v}{\partial z}-2k\nu_1$$
$$=\rho j\left(u\frac{\partial \nu_1}{\partial r}-\frac{1}{r}v\nu_2+w\frac{\partial \nu_1}{\partial z}\right), \tag{3.6}$$

$$\gamma\left[\frac{\partial}{\partial r}\left(\frac{\partial \nu_2}{\partial r}+\frac{\nu_2}{r}\right)+\frac{\partial^2 \nu_2}{\partial z^2}\right]+\kappa\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)-2\kappa\nu_2=\rho j\left(u\frac{\partial \nu_2}{\partial r}+\frac{1}{r}v\nu_1+w\frac{\partial \nu_2}{\partial z}\right), \tag{3.7}$$

and

$$(\alpha+\beta+\gamma)\frac{\partial}{\partial z}\left(\frac{\partial \nu_1}{\partial r}+\frac{\nu_1}{r}+\frac{\partial \nu_3}{\partial z}\right)-\gamma\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial \nu_1}{\partial z}-\frac{\partial \nu_3}{\partial r}\right)\right]+\frac{\kappa}{r}\frac{\partial}{\partial r}(rv)-2\kappa\nu_3$$
$$=\rho j\left(u\frac{\partial \nu_3}{\partial r}+w\frac{\partial \nu_3}{\partial z}\right). \tag{3.8}$$

The boundary conditions are

$$z = 0: \quad u = 0, \quad v = r\Omega, \quad w = 0, \quad \nu_1 = 0, \ \nu_2 = 0, \quad \nu_3 = \Omega, \quad p = 0; \tag{3.9}$$

$$z \to \infty: \quad u = v = 0, \quad \nu_1 = \nu_2 = \nu_3 = 0. \tag{3.10}$$

In order to integrate the equations (3.2) through (3.8), we introduce a dimensionless distance from the disc,

$$\eta = z \sqrt{\frac{\Omega}{k}}, \tag{3.11}$$

where

$$k = \frac{\mu + \kappa}{\rho}. \tag{3.12}$$

Further, using dimensional analysis and keeping in mind the boundary conditions, the velocity components, pressure, and micro-rotation components are found to be of the form:

$$\begin{aligned} &u = r\Omega F_1(\eta), \quad v = r\Omega F_2(\eta), \quad w = \sqrt{k\Omega}\, F_3(\eta), \\ &p = p(z) = -\rho k\Omega P(\eta), \quad \nu_1 = r\Omega \sqrt{\frac{\Omega}{k}}\, G_1(\eta), \\ &\nu_2 = r\Omega \sqrt{\frac{\Omega}{k}}\, G_2(\eta), \quad \nu_3 = \Omega G_3(\eta). \end{aligned} \tag{3.13}$$

If we substitute these forms (3.13) into the equations (3.2) through (3.8), then after some tedious algebra we obtain the following equations:

$$\begin{aligned} &2F_1 + F_3' = 0, \\ &F_1'' - \eta_1 G_2' = F_3 F_1' - F_2^2 + F_1^2, \\ &F_2'' + \eta_1 G_1' = F_3 F_2' + 2F_1 F_2, \\ &F_3'' + 2\eta_1 G_2 + P' = F_3 F_3', \\ &G_1'' - \eta_2 F_2' - 2\eta_2 G_1 = \eta_3 (F_1 G_1 - F_2 G_2 + F_3 G_1'), \\ &G_2'' + \eta_2 F_1' - 2\eta_2 G_2 = \eta_3 (F_1 G_2 + F_2 G_1 + F_3 G_2'), \\ &G_3'' + \eta_4 G_1' + \eta_5 (F_2 - G_3) = \eta_6 F_3 G_3', \end{aligned} \tag{3.14}$$

where the prime denotes differentiation with respect to η, and

$$\begin{aligned} &\eta_1 = \frac{\kappa}{\mu + \kappa}, \quad \eta_2 = \frac{\kappa k}{\gamma \Omega}, \quad \eta_3 = \frac{k\rho j}{\gamma}, \\ &\eta_4 = \frac{2(\alpha + \beta)}{\alpha + \beta + \gamma}, \quad \eta_5 = \frac{2\kappa k}{\Omega(\alpha + \beta + \gamma)}, \quad \eta_6 = \frac{\rho j k}{\alpha + \beta + \gamma}. \end{aligned} \tag{3.15}$$

The constants $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ and η_6 are all dimensionless. The dimensions of the parameters involved are as follows:

$$[\mu, \kappa] = ML^{-1}T^{-1}, \quad [\alpha, \beta, \gamma] = MLT^{-1},$$
$$[j] = L^2, \quad [k] = L^2T^{-1}, \quad [\Omega] = T^{-1},$$
$$[\rho] = ML^{-3}.$$

The boundary conditions (3.9) and (3.10) now take the form

$$\begin{aligned} \eta = 0: \quad & F_1 = 0, \quad F_2 = 1, \quad F_3 = 0, \\ & G_1 = 0, \quad G_2 = 0, \quad G_3 = 1, \quad P = 0; \\ \eta = \infty: \quad & F_1 = F_2 = 0, \quad G_1 = G_2 = G_3 = 0. \end{aligned} \tag{3.16}$$

It is interesting to note that the equations (3.14) reduce to the corresponding equations for a Newtonian fluid [3] for vanishing microrotation (*or* spin) and $\kappa = 0$.

The system of non-linear equations (3.14), subject to the boundary conditions (3.16), does not lend itself to an analytical solution. To solve numerically, it is convenient to reformulate the problem by using the following transformation:

$$\xi = e^{-\eta}. \tag{3.17}$$

Thus the equations (3.14) and boundary conditions (3.16), due to (3.17), become

$$2F_1 - \xi F_3' = 0, \tag{3.18a}$$
$$\xi^2 F_1'' + \xi F_1' + \eta_1 \xi G_2' = -\xi F_3 F_1' - F_2^2 + F_1^2, \tag{3.18b}$$
$$\xi^2 F_2'' + \xi F_2' - \eta_1 \xi G_1' = -\xi F_3 F_2' + 2F_1 F_2, \tag{3.18c}$$
$$2\xi F_1' + 2\eta_1 G_2 - \xi P' = -2F_1 F_3, \tag{3.18d}$$
$$\xi^2 G_1'' + \xi G_1' + \eta_2 \xi F_2' - 2\eta_2 G_1 = \eta_3(F_1 G_1 - F_2 G_2 - \xi F_3 G_1'), \tag{3.18e}$$
$$\xi^2 G_2'' + \xi G_2' - \eta_2 \xi F_1' - 2\eta_2 G_2 = \eta_3(F_1 G_2 + F_2 G_1 - \xi F_3 G_2'), \tag{3.18f}$$
$$\xi^2 G_3'' + \xi G_3' - \eta_4 \xi G_1' + \eta_5(F_2 - G_3) = -\eta_6 \xi F_3 G_3', \tag{3.18g}$$

and

$$\begin{aligned} \xi = 0: \quad & F_1 = F_2 = 0, \quad G_1 = G_2 = G_3 = 0; \\ \xi = 1: \quad & F_1 = 0, \quad F_2 = 1, \quad F_3 = 0, \quad P = 0, \\ & G_1 = G_2 = 0, \quad G_3 = 1, \end{aligned} \tag{3.19}$$

where here the prime denotes differentiation with respect to ξ, and we made use of the first of the equations (3.14) into the fourth of (3.14) which is the equation involving pressure.

The conditions (3.19) are sufficient to obtain a numerical solution of the system of equations (3.18), for given values of the parameters $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ and η_6.

4. Finite-difference equations

We suppose that the interval [0, 1] is divided into grid points with uniform spacing h. We shall denote a typical point by ξ_i. If we then approximate the equations (3.18b, c, e, f, g) by replacing all the derivatives by central – difference approximations, we obtain the following sets of equations:

$$\begin{aligned}(2\xi_i^2 - h\xi_i)F_{i-1}^{(1)} - 4\xi_i^2 F_i^{(1)} + (2\xi_i^2 + h\xi_i)F_{i+1}^{(1)} \\ = -\eta_1 \xi_i h(G_{i+1}^{(2)} - G_{i-1}^{(2)}) - h\xi_i F_i^{(3)}(F_{i+1}^{(1)} - F_{i-1}^{(1)}) \\ - 2h^2 F_i^{(2)} F_i^{(2)} + 2h^2 F_i^{(1)} F_i^{(1)},\end{aligned} \tag{4.1}$$

$$\begin{aligned}(2\xi_i^2 - h\xi_i)F_{i-1}^{(2)} - 4\xi_i^2 F_i^{(2)} + (2\xi_i^2 + h\xi_i)F_{i+1}^{(2)} \\ = \eta_1 \xi_i h(G_{i+1}^{(1)} - G_{i-1}^{(1)}) - \xi_i h F_i^{(3)}(F_{i+1}^{(2)} - F_{i-1}^{(2)}) \\ + 4h^2 F_i^{(1)} F_i^{(2)},\end{aligned} \tag{4.2}$$

$$\begin{aligned}(2\xi_i^2 - h\xi_i)G_{i-1}^{(1)} - (4\xi_i^2 + 4\eta_2 h^2)G_i^{(1)} + (2\xi_i^2 + h\xi_i)G_{i+1}^{(1)} \\ = -\eta_2 \xi_i h(F_{i+1}^{(2)} - F_{i-1}^{(2)}) + \eta_3[2h^2 F_i^{(1)} G_i^{(1)} - 2h^2 F_i^{(2)} G_i^{(2)} \\ - \xi_i h F_i^{(3)}(G_{i+1}^{(1)} - G_{i-1}^{(1)})],\end{aligned} \tag{4.3}$$

$$\begin{aligned}(2\xi_i^2 - h\xi_i)G_{i-1}^{(2)} - 4(\xi_i^2 + \eta_2 h^2)G_i^{(2)} + (2\xi_i^2 + h\xi_i)G_{i+1}^{(2)} \\ = \eta_2 h\xi_i(F_{i+1}^{(1)} - F_{i-1}^{(1)}) + \eta_3[2h^2 F_i^{(1)} G_i^{(2)} + 2h^2 F_i^{(2)} G_i^{(1)} \\ - h\xi_i F_i^{(3)}(G_{i+1}^{(2)} - G_{i-1}^{(2)})],\end{aligned} \tag{4.4}$$

$$\begin{aligned}(2\xi_i^2 - h\xi_i)G_{i-1}^{(3)} - 2(2\xi_i^2 + \eta_5 h^2)G_i^{(3)} + (2\xi_i^2 + h\xi_i)G_{i+1}^{(3)} \\ = \eta_4 h\xi_i(G_{i+1}^{(1)} - G_{i-1}^{(1)}) \\ - 2\eta_5 h^2 F_i^{(2)} - \eta_6 \xi_i h F_i^{(3)}(G_{i+1}^{(3)} - G_{i-1}^{(3)}),\end{aligned} \tag{4.5}$$

which must be solved for all grid points interior to the interval [0, 1] using the conditions (3.19), for F_1, F_2, G_1, G_2 and G_3 at the boundary points.

Equations (3.18a) and (3.18d) will be integrated using the boundary conditions on F_3 and P.

5. Computational procedure

The ultimate object is to solve the above system of finite-difference equations at each required grid point. The sets of equations (4.1) through (4.5) are solved, in turn, subject to the appropriate boundary conditions, by the Gauss-Seidel iterative procedure [5, p. 154 – 156], whereas the equations (3.18a) and (3.18d) are integrated using the Simpson's rule [5, p. 67 – 69] with the formula given in [9, p. 48] and the conditions on F_3 and P.

The overall iterative sequence is as follows:

(i) The set of equations (4.1) is solved subject to $F_1 = 0$ when $\xi = 0$ and $\xi = 1$, making use of the most recently available information for F_1, F_2, F_3 and G_2. Then the equation (3.18a) is integrated using Simpson's rule with $F_3 = 0$ when $\xi = 1$.

(ii) The computed solution is introduced into (4.2) and this set of equations is solved subject to $F_2 = 0$ when $\xi = 0$ and $F_2 = 1$ when $\xi = 1$. The most recently available values of F_2, F_3 and G_1 are again used to calculate the new values of F_2.

(iii) The equations (3.18a), (4.3), (4.4) and (4.5) are solved similarly, with the appropriate boundary conditions for the corresponding functions, using the most recent values for F_1, F_2, F_3, G_1, G_2 and G_3. The solution for these equations provides updated information for the subsequent application of (i) and (ii).

The above round-off procedure is repeated as many times as necessary until all the solutions have approached definite limits to some required criterion of accuracy; this defines the notion of convergence. The criterion $|F_i^{(n+1)} - F_i^{(n)}| < 10^{-6}$, $|G_i^{(n+1)} - G_i^{(n)}| < 10^{-6}$, where $i = 1$, 2, 3, was adopted as a terminating condition and here the subscript denotes the number of a function, and superscript the iteration.

The computed solutions of F_1, F_3 and G_2 were employed to solve (3.18d), that is, to calculate the pressure P with the condition $P = 0$ when $\xi = 1$.

It is pointed out that to calculate the values of F_3 and P at the point $\xi = 0$ only, the first derivative was approximated by forward-differences, after the application of the l'Hôpital's rule.

6. Results

The calculations were carried out for the following four cases:

	η_1	η_2	η_3	η_4	η_5	η_6
a)	0.1	0.4	0.2	0.3	0.4	0.1
b)	0.4	0.9	0.2	0.1	1.0	0.4
c)	0.3	0.8	0.2	0.1	0.6	0.3
d)	0.2	0.6	0.1	0.2	0.8	0.1

In Tables 1, 2, 3, and 4, the values of $F_1, F_2, F_3, G_1, G_2, G_3$ and P are shown for the cases considered. The accuracy of the results was checked by comparing with different grid sizes, and with the results given in [3] for the usual Newtonian fluids. The effect of the microrotation is noticed for all the four cases, as expected. For the case (a), Figures 2 and 3 indicate that F_1, F_2, F_3 and P are slightly less than that of the Newtonian case.

TABLE 1

Case (a)

ξ	F_1	F_2	F_3	G_1	G_2	G_3	P
0.000	0.000000	0.000000	−0.863057	0.000000	0.000000	0.000000	0.385344
0.025	0.031068	0.042498	−0.797701	0.027651	−0.013361	0.139119	0.389314
0.050	0.056008	0.081070	−0.738820	0.042742	−0.017961	0.217077	0.391780
0.075	0.077190	0.117600	−0.685174	0.053530	−0.019999	0.278128	0.394402
0.100	0.095457	0.152690	−0.635631	0.061715	−0.020688	0.329718	0.397045
0.125	0.111286	0.186634	−0.589545	0.068061	−0.020555	0.374926	0.399548
0.150	0.124993	0.219602	−0.546482	0.073002	−0.019886	0.415412	0.401762
0.175	0.136814	0.251708	−0.506125	0.076823	−0.018860	0.452200	0.403566
0.200	0.146933	0.283028	−0.468230	0.079716	−0.017594	0.485981	0.404855
0.225	0.155500	0.313621	−0.432599	0.081827	−0.016172	0.517246	0.405542
0.250	0.162639	0.343530	−0.399069	0.083265	−0.014652	0.54366	0.405551
0.275	0.168457	0.372791	−0.367501	0.084118	−0.013081	0.573624	0.404823
0.300	0.173048	0.401429	−0.337775	0.084455	−0.011492	0.599246	0.403309
0.325	0.176492	0.429470	−0.309785	0.084336	−0.009912	0.623415	0.400963
0.350	0.178864	0.456931	−0.283440	0.083807	−0.008362	0.646283	0.397757
0.375	0.180226	0.483830	−0.258655	0.082912	−0.006858	0.667975	0.393658
0.400	0.180639	0.510181	−0.235356	0.081684	−0.005414	0.688599	0.388650
0.425	0.180156	0.535997	−0.213474	0.080156	−0.004039	0.708248	0.382716
0.450	0.178824	0.561289	−0.192946	0.078353	−0.002743	0.726999	0.375842
0.475	0.176690	0.586068	−0.173717	0.076301	−0.001533	0.744924	0.368025
0.500	0.173795	0.610342	−0.155731	0.074018	−0.000413	0.762084	0.359263
0.525	0.170175	0.634120	−0.138942	0.071526	+0.000611	0.778532	0.349550
0.550	0.165868	0.657411	−0.123302	0.068841	+0.001536	0.794317	0.338895
0.575	0.160905	0.680221	−0.108770	0.065977	+0.002359	0.809483	0.327300
0.600	0.155317	0.702557	−0.095306	0.062949	+0.003079	0.824069	0.314774
0.625	0.149134	0.724426	−0.082872	0.059770	+0.003693	0.838109	0.301328
0.650	0.142381	0.745834	−0.071433	0.056451	+0.004200	0.851636	0.286970
0.675	0.135085	0.766787	−0.060957	0.053003	+0.004599	0.864680	0.271718
0.700	0.127268	0.787289	−0.051411	0.049435	+0.004890	0.877267	0.255582
0.725	0.118955	0.807348	−0.042766	0.045756	+0.005073	0.889421	0.238584
0.750	0.110165	0.826967	−0.034994	0.041975	+0.005147	0.901165	0.220737
0.775	0.100918	0.846152	−0.028069	0.038100	+0.005112	0.912521	0.202058
0.800	0.091234	0.864907	−0.021964	0.034137	+0.004970	0.923506	0.182570
0.825	0.081131	0.883237	−0.016656	0.030094	+0.004719	0.934140	0.162291
0.850	0.070625	0.901147	−0.012123	0.025975	+0.004361	0.944439	0.141241
0.875	0.059734	0.918640	−0.008340	0.021788	+0.003897	0.954419	0.119443
0.900	0.048471	0.935720	−0.005289	0.017538	+0.003326	0.964093	0.096918
0.925	0.036853	0.952393	−0.002948	0.013229	+0.002650	0.973476	0.073689
0.950	0.024893	0.968661	−0.001299	0.008867	+0.001870	0.982581	0.049777
0.975	0.012604	0.984529	−0.000322	0.004456	+0.000986	0.991418	0.025206
1.000	0.000000	1.000000	0.000000	0.000000	+0.000000	1.000000	0.000000

TABLE 2

Case (b)

ξ	F_1	F_2	F_3	G_1	G_2	G_3	P
0.000	0.000000	0.000000	−0.850417	0.000000	0.000000	0.000000	0.386363
0.025	0.029345	0.038746	−0.789656	0.028318	−0.013972	0.092658	0.387922
0.050	0.053986	0.076318	−0.733406	0.046177	−0.019275	0.158432	0.389749
0.075	0.075182	0.112568	−0.681407	0.059654	−0.021658	0.214008	0.392022
0.100	0.093587	0.147701	−0.632986	0.070281	−0.022431	0.263126	0.394463
0.125	0.109604	0.181858	−0.587695	0.078804	−0.022193	0.307537	0.396856
0.150	0.123520	0.215134	−0.545208	0.085667	−0.021288	0.348260	0.399031
0.175	0.135550	0.247599	−0.505274	0.091167	−0.019931	0.385966	0.400843
0.200	0.145871	0.279306	−0.467690	0.095517	−0.018269	0.421129	0.402178
0.225	0.154626	0.310296	−0.432287	0.098877	−0.016408	0.454102	0.402939
0.250	0.161937	0.340603	−0.398923	0.101373	−0.014425	0.485155	0.403044
0.275	0.167911	0.370252	−0.367474	0.103106	−0.012378	0.514508	0.402429
0.300	0.172640	0.399268	−0.337830	0.104157	−0.010312	0.542335	0.401038
0.325	0.176206	0.427669	−0.309896	0.104596	−0.008264	0.568785	0.398826
0.350	0.178682	0.455472	−0.283586	0.104482	−0.006260	0.593980	0.395755
0.375	0.180133	0.482693	−0.258820	0.103865	−0.004324	0.618026	0.391794
0.400	0.180620	0.509344	−0.235528	0.102789	−0.002474	0.641015	0.386923
0.425	0.180197	0.535438	−0.213644	0.101292	−0.000723	0.663026	0.381123
0.450	0.178914	0.560985	−0.193109	0.099408	+0.000915	0.684130	0.374383
0.475	0.176816	0.585996	−0.173868	0.097168	+0.002432	0.704387	0.366693
0.500	0.173947	0.610479	−0.155868	0.094597	+0.003819	0.723855	0.358052
0.525	0.170345	0.634444	−0.139063	0.091721	+0.005069	0.742582	0.348456
0.550	0.166047	0.657898	−0.123407	0.088562	+0.006179	0.760613	0.337910
0.575	0.161086	0.680849	−0.108859	0.085140	+0.007144	0.777990	0.326418
0.600	0.155495	0.703304	−0.095380	0.081473	+0.007959	0.794748	0.313988
0.625	0.149303	0.725270	−0.082931	0.077577	+0.008624	0.810923	0.300629
0.650	0.142538	0.746754	−0.071480	0.073469	+0.009136	0.826544	0.286355
0.675	0.135227	0.767761	−0.060992	0.069162	+0.009493	0.841641	0.271179
0.700	0.127393	0.788299	−0.051436	0.064670	+0.009694	0.856240	0.255113
0.725	0.119060	0.808372	−0.042783	0.060004	+0.009740	0.870365	0.238176
0.750	0.110250	0.827986	−0.035005	0.055176	+0.009628	0.884039	0.220385
0.775	0.100984	0.847147	−0.028074	0.050197	+0.009361	0.897283	0.201760
0.800	0.091282	0.865859	−0.021966	0.045076	+0.008937	0.910117	0.182319
0.825	0.081161	0.884129	−0.016656	0.039822	+0.008357	0.922559	0.162082
0.850	0.0706406	0.901960	−0.012121	0.034445	+0.007622	0.934626	0.141071
0.875	0.059736	0.919357	−0.008338	0.028952	+0.006733	0.946335	0.119307
0.900	0.048465	0.936326	−0.005287	0.023350	+0.005690	0.957700	0.096814
0.925	0.036841	0.952870	−0.002947	0.017648	+0.004494	0.968736	0.073613
0.950	0.024880	0.968994	−0.001298	0.011851	+0.003146	0.979456	0.049728
0.975	0.012595	0.984703	−0.000322	0.005967	+0.001648	0.989874	0.025182
1.000	0.000000	1.000000	0.000000	0.000000	+0.000000	1.000000	0.000000

TABLE 3

Case (c)

ξ	F_1	F_2	F_3	G_1	G_2	G_3	P
0.000	0.000000	0.000000	−0.834636	0.000000	0.000000	0.000000	0.387254
0.025	0.027344	0.035252	−0.778923	0.025432	−0.013816	0.092195	0.386922
0.050	0.051464	0.071485	−0.725843	0.044210	−0.020279	0.158158	0.388146
0.075	0.072570	0.107166	−0.675941	0.059236	−0.023622	0.214035	0.390022
0.100	0.091074	0.142124	−0.629002	0.071579	−0.025094	0.263474	0.392185
0.125	0.107283	0.176337	−0.584794	0.081822	−0.025326	0.308188	0.394387
0.150	0.121433	0.209814	−0.543113	0.090336	−0.024694	0.349183	0.396437
0.175	0.133715	0.242573	−0.503785	0.097383	−0.023450	0.387122	0.398183
0.200	0.144287	0.274632	−0.466658	0.103160	−0.021772	0.422478	0.399496
0.225	0.153285	0.306010	−0.431600	0.107818	−0.019790	0.455602	0.400275
0.250	0.160825	0.336723	−0.398494	0.111479	−0.017603	0.486771	0.400431
0.275	0.167008	0.366788	−0.367237	0.114246	−0.015286	0.516205	0.399888
0.300	0.171927	0.396218	−0.337734	0.116202	−0.012902	0.544082	0.398589
0.325	0.175663	0.425027	−0.309901	0.117419	−0.010498	0.570554	0.396486
0.350	0.178287	0.453227	−0.283659	0.117960	−0.008114	0.595746	0.393532
0.375	0.179868	0.480828	−0.258939	0.117880	−0.005782	0.619768	0.389699
0.400	0.180465	0.507841	−0.235674	0.117226	−0.003529	0.642714	0.384959
0.425	0.180135	0.534277	−0.213803	0.116041	−0.001377	0.664667	0.379295
0.450	0.178928	0.569143	−0.193271	0.114364	+0.000656	0.685698	0.372689
0.475	0.176891	0.585450	−0.174025	0.112229	+0.002554	0.705873	0.365133
0.500	0.174068	0.610206	−0.156015	0.109666	+0.004306	0.725250	0.356621
0.525	0.170499	0.634418	−0.139196	0.106705	+0.005899	0.743879	0.347150
0.550	0.166224	0.658094	−0.123525	0.103372	+0.007326	0.761809	0.336726
0.575	0.161276	0.681241	−0.108960	0.099689	+0.008578	0.779080	0.325349
0.600	0.155689	0.703867	−0.095465	0.095679	+0.009650	0.795733	0.313029
0.625	0.149493	0.725978	−0.083001	0.091363	+0.010536	0.811801	0.299772
0.650	0.142719	0.747582	−0.071534	0.086758	+0.011231	0.827319	0.285595
0.675	0.135392	0.768683	−0.061033	0.081883	+0.011734	0.842314	0.270505
0.700	0.127540	0.789290	−0.051466	0.076753	+0.012040	0.856816	0.254522
0.725	0.119187	0.809407	−0.042804	0.071383	+0.012148	0.870848	0.237663
0.750	0.110354	0.829041	−0.035018	0.065788	+0.012055	0.884436	0.219941
0.775	0.101065	0.848198	−0.028081	0.059981	+0.011761	0.897600	0.201378
0.800	0.091340	0.866882	−0.021969	0.053974	+0.011265	0.910361	0.181995
0.825	0.081199	0.885100	−0.016656	0.047780	+0.010566	0.922739	0.161812
0.850	0.070659	0.902857	−0.012119	0.041408	+0.009664	0.934750	0.140849
0.875	0.059739	0.920158	−0.008335	0.034869	+0.008558	0.946412	0.119129
0.900	0.048456	0.937008	−0.005284	0.028174	+0.007250	0.957741	0.096676
0.925	0.036826	0.953413	−0.002945	0.021331	+0.005740	0.968751	0.073513
0.950	0.024863	0.969376	−0.001297	0.014348	+0.004027	0.979455	0.049663
0.975	0.012583	0.984904	−0.000321	0.007236	+0.002114	0.989867	0.025150
1.000	0.000000	1.000000	0.000000	0.000000	+0.000000	1.000000	0.000000

TABLE 4

Case (d)

ξ	F_1	F_2	F_3	G_1	G_2	G_3	P
0.000	0.000000	0.000000	−0.819329	0.000000	0.000000	0.000000	0.387268
0.025	0.025399	0.031481	−0.768541	0.024280	−0.013573	0.062790	0.385036
0.050	0.049025	0.066355	−0.718534	0.043667	−0.020489	0.118187	0.385689
0.075	0.070050	0.101488	−0.670658	0.059537	−0.024212	0.168576	0.387211
0.100	0.088655	0.136306	−0.625147	0.072763	−0.025954	0.215085	0.389140
0.125	0.105052	0.170611	−0.581980	0.083862	−0.026354	0.258401	0.391196
0.150	0.119430	0.204322	−0.541073	0.093178	−0.025806	0.298994	0.393166
0.175	0.131955	0.237403	−0.502326	0.100965	−0.024577	0.337213	0.394887
0.200	0.142771	0.269841	−0.465637	0.107412	−0.022856	0.373328	0.396223
0.225	0.152002	0.301629	−0.430909	0.112671	−0.020787	0.407556	0.397058
0.250	0.159760	0.332770	−0.398051	0.116866	−0.018477	0.440075	0.397297
0.275	0.166145	0.363266	−0.366978	0.120098	−0.016012	0.471035	0.396864
0.300	0.171245	0.393126	−0.337610	0.122455	−0.013460	0.500565	0.395690
0.325	0.175142	0.422354	−0.309872	0.124012	−0.010875	0.528777	0.393724
0.350	0.177908	0.450959	−0.283698	0.124833	−0.008303	0.555768	0.390918
0.375	0.179611	0.478949	−0.259021	0.124976	−0.005779	0.581623	0.387239
0.400	0.180312	0.506331	−0.235782	0.124490	−0.003334	0.606421	0.382656
0.425	0.180068	0.533113	−0.213925	0.123422	−0.000994	0.630229	0.377148
0.450	0.178932	0.559303	−0.193396	0.121812	+0.001221	0.653109	0.370701
0.475	0.176950	0.584909	−0.174146	0.119696	+0.003292	0.675118	0.363297
0.500	0.174170	0.609939	−0.156128	0.117107	+0.005204	0.696307	0.354936
0.525	0.170632	0.634399	−0.139298	0.114077	+0.006945	0.716723	0.345611
0.550	0.166375	0.658298	−0.123613	0.110633	+0.008504	0.736408	0.335323
0.575	0.161438	0.681642	−0.109035	0.106801	+0.009872	0.755403	0.324081
0.600	0.155852	0.704438	−0.095525	0.102604	+0.011402	0.773743	0.311883
0.625	0.149652	0.726694	−0.083048	0.098065	+0.012008	0.791463	0.298746
0.650	0.142867	0.748416	−0.071569	0.093203	+0.012764	0.808594	0.284678
0.675	0.135526	0.769611	−0.061058	0.088038	+0.013305	0.825166	0.269693
0.700	0.127656	0.790286	−0.051481	0.082587	+0.013630	0.841204	0.253806
0.725	0.119282	0.810447	−0.042812	0.076866	+0.013734	0.856736	0.237034
0.750	0.110428	0.830099	−0.035020	0.070892	+0.013615	0.871784	0.219394
0.775	0.101117	0.849250	−0.028079	0.064679	+0.013272	0.886371	0.200907
0.800	0.091372	0.867906	−0.021964	0.058240	+0.012704	0.900518	0.181593
0.825	0.081211	0.886072	−0.016650	0.051588	+0.011909	0.914245	0.161473
0.850	0.070655	0.903754	−0.012112	0.044735	+0.010888	0.927570	0.140569
0.875	0.059723	0.920958	−0.008330	0.037693	+0.009639	0.940511	0.118904
0.900	0.048432	0.937689	−0.005280	0.030472	+0.008163	0.953084	0.096502
0.925	0.036798	0.953954	−0.002942	0.023083	+0.006461	0.965305	0.073385
0.950	0.024838	0.969757	−0.001295	0.015535	+0.004533	0.977189	0.049580
0.975	0.012567	0.985104	−0.000321	0.007838	+0.002379	0.988749	0.025110
1.000	0.000000	1.000000	0.000000	0.000000	+0.000000	1.000000	0.000000

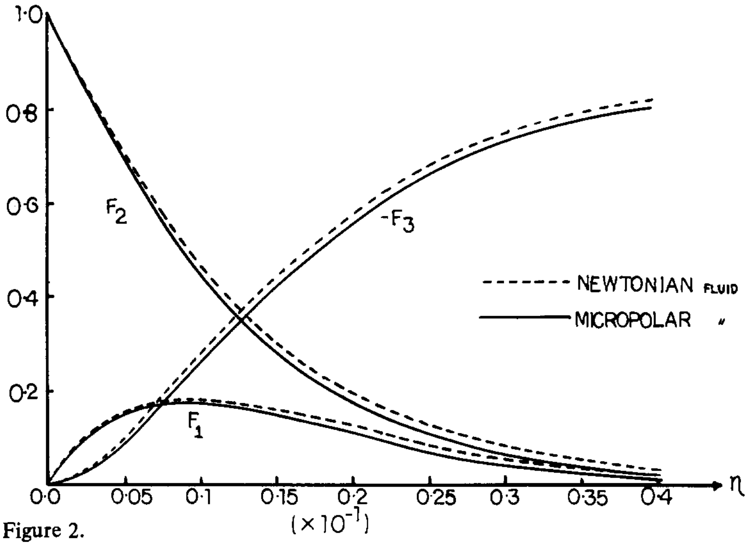

Figure 2.

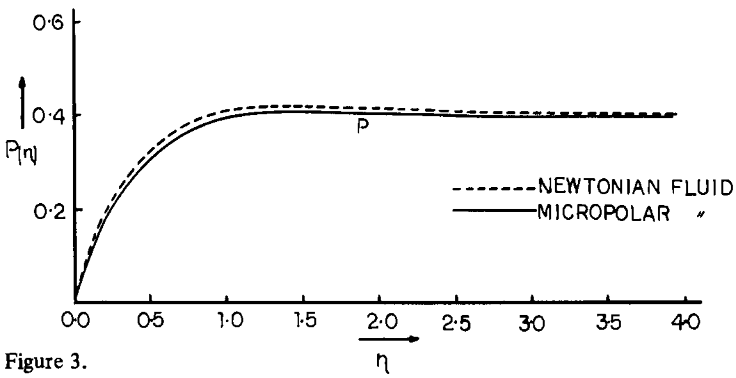

Figure 3.

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